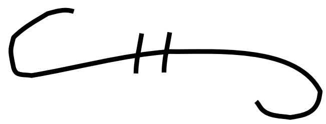
Special Vector Calculus Session
For
Engineering Electromagnetics I



by

Professor Robert A. Schill Jr.

Special Vector Calculus Session for Engineering Electromagnetics I

1. Simple computation of a curl, divergence and gradient of vector $\vec{F} = r\cos\theta\hat{r}(\theta,\varphi)$. [Spherical Coordinate System]

Curl

$$\vec{\nabla}\times\vec{F} = \frac{1}{r\sin\theta}\left[\frac{\partial(F_\varphi\sin\theta)}{\partial\theta} - \frac{\partial(F_\theta)}{\partial\varphi}\right]\hat{r}(\theta,\varphi) +$$

$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial(rF_\varphi)}{\partial r}\right]\hat{\theta}(\theta,\varphi)$$

$$+\frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\hat{\varphi}(\varphi)$$

$$= -\frac{1}{r}\frac{\partial F_r}{\partial\theta}\hat{\varphi}(\varphi) = \sin\theta\,\hat{\varphi}(\varphi)$$

Divergence

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[F_\theta\sin\theta\right] + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

$$= \frac{1}{r^2}\cos\theta\frac{\partial}{\partial r}r^3 = 3\cos\theta$$

Gradient [Careful, this is a gradient of a vector.]

$$\vec{\nabla}\vec{F} = \vec{\nabla}[r\cos\theta\,\hat{r}(\theta,\varphi)]$$

$$= \hat{r}\frac{\partial}{\partial r}[r\cos\theta\,\hat{r}(\theta,\varphi)] + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}[r\cos\theta\,\hat{r}(\theta,\varphi)] +$$

$$+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left[r\cos\theta\,\hat{r}(\theta,\varphi)\right]$$

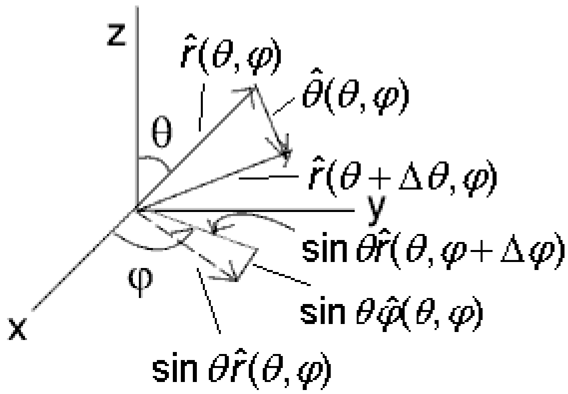
$$= \cos\theta\,\hat{r}(\theta,\varphi)\hat{r}(\theta,\varphi) + \hat{\theta}(\theta,\varphi)[-\sin\theta\,\hat{r}(\theta,\varphi) +$$

$$+\cos\frac{\partial}{\partial\theta}\hat{r}(\theta,\varphi)] + \hat{\varphi}(\varphi)\cot\theta\frac{\partial}{\partial\theta}\hat{r}(\theta,\varphi)$$

$$\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\frac{\partial}{\partial\theta}\hat{r}(\theta,\varphi) = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z}$$

$$= \hat{\theta}(\theta,\varphi)$$



$$\frac{\partial}{\partial\varphi}\hat{r}(\theta,\varphi) = -\sin\theta\sin\varphi\,\hat{x} + \sin\theta\cos\varphi\,\hat{y}$$

$$= \sin\theta\,\hat{\varphi}(\varphi)$$

$$\therefore$$

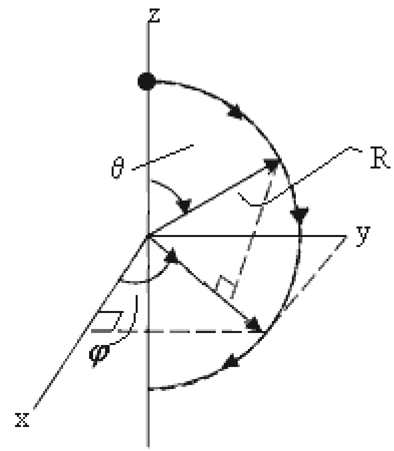
$$\vec{\nabla}\vec{F} = \cos\theta\,\hat{r}(\theta,\varphi)\hat{r}(\theta,\varphi) - \sin\theta\,\hat{\theta}(\theta,\varphi)\hat{r}(\theta,\varphi) +$$

$$+\cos\theta\,\hat{\theta}(\theta,\varphi)\hat{\theta}(\theta,\varphi) + \cos\theta\,\hat{\varphi}(\varphi)\hat{\varphi}(\varphi)$$

2. Integration of a vector over a coordinate.

$$\int_o^{\pi} \vec{F} d\theta \Big|_{\substack{r=R \\ \varphi=\varphi_o}} \quad \text{where } \vec{F} = x\hat{r}(\theta,\varphi) \text{ (mixed coordinate system)}$$

In this mixed coordinate system, the coefficient x is a function of r,θ and φ.
Referring to the figure below, as θ changes, x changes.
From the figure,



$$x = r\sin\theta\cos\varphi \Big|_{\substack{r=R \\ \varphi=\varphi_o}}$$

$$\hat{r}(\theta,\varphi) = \hat{x}\sin\theta\cos\theta + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta\Big|_{\varphi=\varphi_o}$$

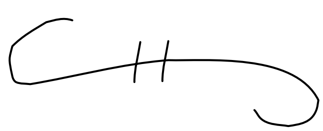
$\therefore$

$$\vec{F} = R\left[\hat{x}\sin^2\theta\cos^2\varphi_o + \hat{y}\sin^2\theta\cos\varphi_o\sin\varphi_o + \hat{z}\cos\theta\sin\theta\cos\varphi_o\right]$$

$$\int_o^{\pi} \vec{F} d\theta \Big|_{\substack{r=R \\ \varphi=\varphi_o}} = R\int_o^{\pi}\left[\hat{x}\sin^2\theta\cos^2\varphi_o + \hat{y}\sin^2\theta\cos\varphi_o\sin\varphi_o + \hat{z}\cos\theta\sin\theta\cos\varphi_o\right]d\theta$$

$$= \hat{x}R\cos^2\varphi_o\int_o^{\pi}\sin^2\theta d\theta + \hat{y}R\cos\varphi_o\sin\varphi_o\int_o^{\pi}\sin^2\theta d\theta +$$

$$+ \hat{z}R\cos\varphi_o\int_o^{\pi}\cos\theta\sin\theta d\theta$$



3. Cross product of vectors: $\vec{A}\times\vec{B}$

a) $\vec{A} = A_r\hat{r}(\varphi_A)$ $\quad\quad$ $\vec{B} = B_r\hat{r}(\varphi_B)$

Both in a cylindrical coordinate system

$$\vec{A}\times\vec{B} = A_r B_r\left[\hat{r}(\varphi_A)\times\hat{r}(\varphi_B)\right]$$

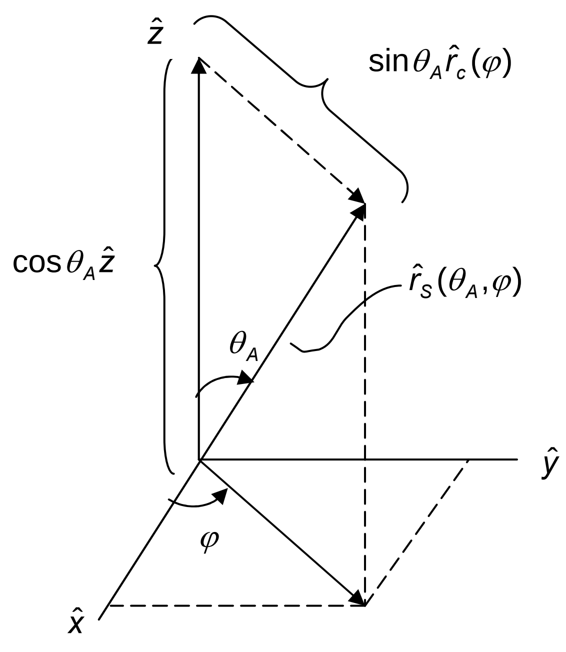
But

$$\hat{r}(\varphi_A) = \cos\varphi_A\hat{x} + \sin\varphi_A\hat{y}$$
$$\hat{r}(\varphi_B) = \cos\varphi_B\hat{x} + \sin\varphi_B\hat{y}$$

$$\vec{A}\times\vec{B} = A_r B_r\left[\cos\varphi_A\hat{x} + \sin\varphi_A\hat{y}\right]\times\left[\cos\varphi_B\hat{x} + \sin\varphi_B\hat{y}\right]$$
$$= A_r B_r\left[\cos\varphi_A\sin\varphi_B - \sin\varphi_A\cos\varphi_B\right]\hat{z}$$
$$= A_r B_r\sin(\varphi_B - \varphi_A)\hat{z}$$

b) $\vec{A} = A_r\hat{r}_s(\theta_A,\varphi)$ (spherical) $\quad$ $\vec{B} = B_r\hat{r}_c(\varphi)$ $\quad\quad$ (cylindrical)

Re-express the $\vec{A}$ vector from a spherical to a cylindrical coordinate system.



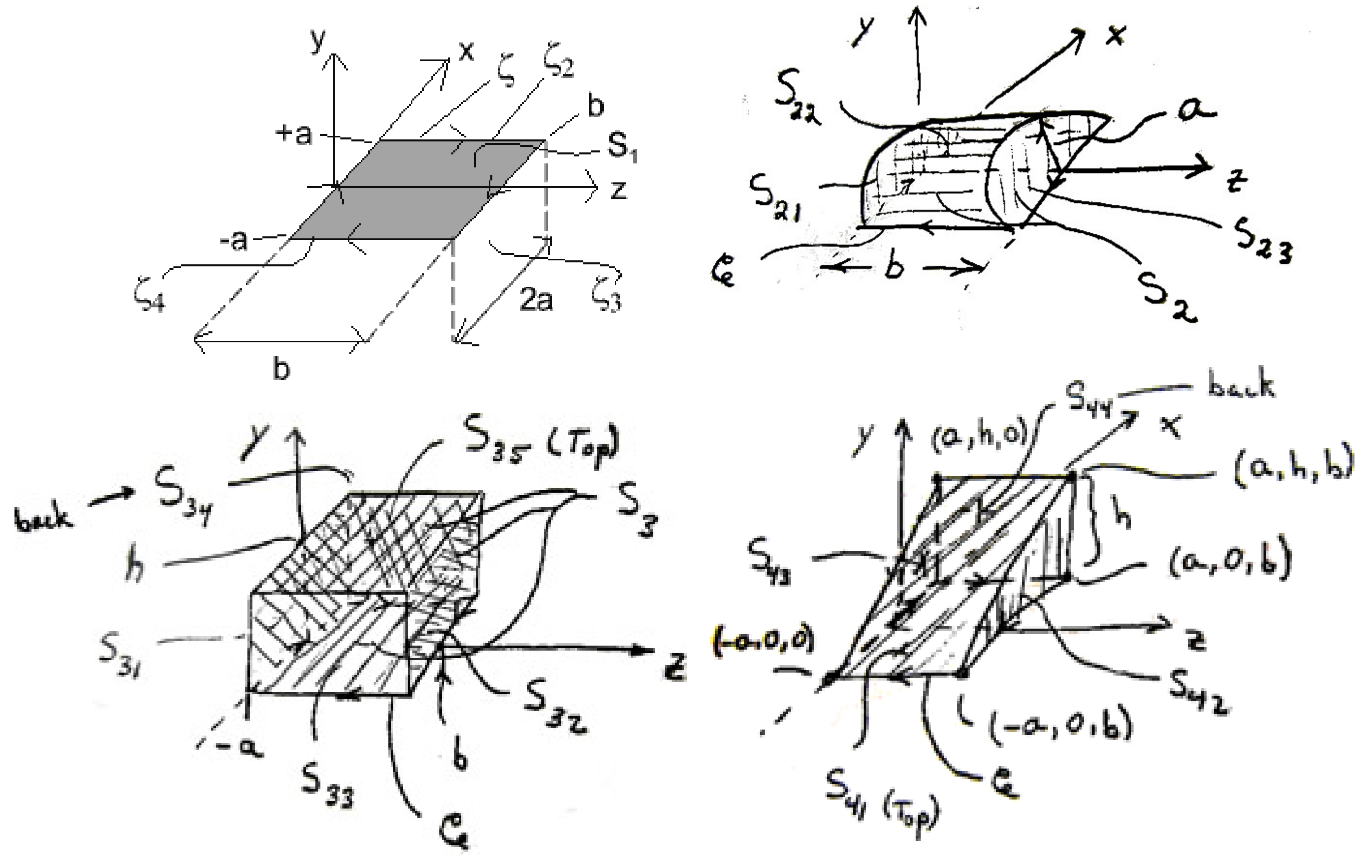
$\therefore$

$$\hat{r}_s(\theta_A,\varphi) = \sin\theta_A \hat{r}_c(\varphi) + \cos\theta_A \hat{z}$$

$$\therefore \vec{A}\times\vec{B} = A_r \hat{r}_s(\theta_A,\varphi)\times B_r \hat{r}_c(\varphi) = A_r B_r \left[(\sin\theta_A \hat{r}_c(\varphi) + \cos\theta_A \hat{z})\times \vec{r}_c(\varphi)\right]$$

$$= A_r B_r \cos\theta_A \hat{\varphi}(\varphi)$$

4. Verify Stoke's Theorem for the function $\vec{F} = (x + y + z)\hat{z}$ along the contour and over the bounded surfaces below.



Stoke's Theorem

$$\oint_{c} \vec{F} \cdot d\vec{l} = \iint_{s} (\vec{\nabla} \times \vec{F}) \cdot d\vec{s}$$

Note the curl is evaluated over the surface S bounded by ζ using a RHR convention.

No other constraints are imposed on the surface S.

<u>Along Contour ζ</u>

$$\oint_{\zeta} \vec{F} \cdot d\vec{l} = \int_{\zeta_1} \vec{F} \cdot d\vec{l}_1 + \int_{\zeta_2} \vec{F} \cdot d\vec{l}_2 + \int_{\zeta_3} \vec{F} \cdot d\vec{l}_3 + \int_{\zeta_4} \vec{F} \cdot d\vec{l}_4$$

$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$ $d\vec{l}$ is *ALWAYS* positive, by convention, signs are incorporated in the limits of integration.

$\therefore$

$d\vec{l} = {}^{+}dx\hat{x}, \quad d\vec{l}_2 = {}^{+}dz\hat{z}, \quad d\vec{l}_3 = {}^{+}dx\hat{x}, \quad d\vec{l}_4 = {}^{+}dz\hat{z}$

The contour ζ lies in the y = 0 plane. Thus,

$$\oint_{\zeta} \vec{F}\cdot d\vec{l} = \int_{x=-a}^{a} (x+y+z)\hat{z}\cdot dx\hat{x}\ \Big|_{\substack{y=0\\z=0}} + \int_{z=o}^{b} (x+y+z)\hat{z}\cdot dz\hat{z}\ \Big|_{\substack{x=a\\y=0}} +$$

$$+\int_{x=a}^{-a} (x+y+z)\hat{z}\cdot dx\hat{x}\ \Big|_{\substack{y=0\\z=b}} + \int_{z=b}^{o} (x+y+z)\hat{z}\cdot dz\hat{z}\ \Big|_{\substack{y=0\\x=-a}}$$

$$= 0 + \int_{o}^{b} (a+z)dz + 0 + \int_{b}^{o} (-a+z)dz$$

$$= \left[az + \frac{z^2}{2} \right|_{o}^{b} + \left[-az + \frac{z^2}{2} \right|_{b}^{o}$$

$$= \left[az + \frac{z^2}{2} \right|_{o}^{b} - \left[-az + \frac{z^2}{2} \right|_{0}^{b} = 2ab$$

$\therefore$

$$\oint_{\zeta} \vec{F}\cdot d\vec{l} = 2ab$$

<u>Over Surface S_1</u>

By the RHR, $d\vec{S}_1$ points in $-\hat{y}$ direction

$$d\vec{S}_1 = -\hat{y}dxdz$$

$$\vec{\nabla}\times\vec{F} = \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{x} + \left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right)\hat{y} + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{z}$$

$$= \hat{x} - \hat{y}$$

$$\iint_{S_1} \left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{s} = \int_{x=-a}^{a}\int_{z=0}^{b} [\hat{x}-\hat{y}]\cdot\left(-\hat{y}dxdz\right)\ \Big|_{y=0} =$$

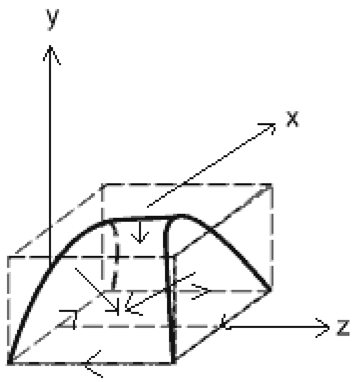
$$= \int_{-a}^{a}\int_{0}^{b} dxdz\ \Big|_{y=0} = 2ab$$

Over Surface S_2

By the RHR, $d\vec{S}_{22}$ points radially inward in a cylindrical coordinate system.

$$d\vec{S}_{22} = -\hat{r}(\varphi) dS_{22}$$

Allowing the two ends of the cylinder to be slanted slightly inward such that the y = a points of the half cylinder lies between 0<z<b it is clear by the RHR



$$d\vec{S}_{21} = \hat{z} dS_{21} \Big|_{\substack{z\to 0 \\ plane}} \text{ and } d\vec{S}_{23} = -\hat{z} dS_{23} \Big|_{z\to b\ plane}$$

$$\vec{\nabla}\times\vec{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right)\hat{r} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\varphi} + \frac{1}{r}\left[\frac{\partial (rF_\varphi)}{\partial r} - \frac{\partial F_r}{\partial \varphi}\right]\hat{z}$$

Transform $\vec{F}$ from a Cartesian to a cylindrical coordinate system.

Cartesian (x, y, z)		Cylindrical (r,φ,z)
$\hat{z}$	⇌	$\hat{z}$
x	⇌	rcosφ
y	⇌	rsinφ
z	⇌	z

$\therefore$

$$\vec{F} = [r\cos\varphi + r\sin\varphi + z]\hat{z}$$

$$\vec{\nabla}\times\vec{F} = \frac{1}{r}\frac{\partial F_z}{\partial \varphi}\hat{r}(\varphi) - \frac{\partial F_z}{\partial r}\hat{\varphi}(\varphi)$$

$$= [-\sin\varphi + \cos\varphi]\hat{r}(\varphi) - [\cos\varphi + \sin\varphi]\hat{\varphi}(\varphi)$$

$\therefore$

$$\iint_{S_2}(\vec{\nabla}\times\vec{F})\cdot d\vec{s} = \int_{r=0}^{a}\int_{\varphi=0}^{\pi}\{[-\sin\varphi+\cos\varphi]\hat{r}-[\cos\varphi+\sin\varphi]\hat{\varphi}\}\cdot\hat{z}rdrd\varphi\Big|_{z=0}+$$

$$+\int_{r=0}^{a}\int_{\varphi=0}^{\pi}\{[-\sin\varphi+\cos\varphi]\hat{r}-[\cos\varphi+\sin\varphi]\hat{\varphi}\}\cdot(-\hat{z}rdrd\varphi)\Big|_{z=b}+$$

$$+\int_{\varphi=0}^{\pi}\int_{z=0}^{b}\{[-\sin\varphi+\cos\varphi]\hat{r}-[\cos\varphi+\sin\varphi]\hat{\varphi}\}\cdot(-\hat{r}(\varphi)rd\varphi dz)\Big|_{r=a}$$

$$=0+0+\int_{\varphi=0}^{\pi}\int_{z=0}^{b}[\sin\varphi-\cos\theta]ad\varphi dz$$

$$=ab\int_0^{\pi}(\sin\varphi-\cos\varphi)d\varphi = ab[-\cos\varphi-\sin\varphi\Big|_0^{\pi}$$

$$=ab[-(-1)-0-\{-(+1)-0\}]=2ab$$

<u>Over Surface S_3</u>

Similar to integrations over S_1 and S_2. By arguments given there, we may write,

$$d\vec{S}_{31}=+\hat{z}dxdy\Big|_{z=0}\quad d\vec{S}_{32}=-\hat{z}dxdy\Big|_{z=b}$$

$$d\vec{S}_{33}=+\hat{x}dydz\Big|_{x=-a}\quad d\vec{S}_{34}=-\hat{x}dydz\Big|_{x=+a}$$

$$d\vec{S}_{35}=-\hat{y}dxdz\Big|_{y=h}$$

$$\vec{\nabla}\times\vec{F}=\hat{x}-\hat{y}$$

$$\therefore\iint_{S_3}(\vec{\nabla}\times\vec{F})\cdot d\vec{s}=\underbrace{\int_{x=-a}^{a}\int_{y=0}^{h}(\hat{x}-\hat{y})\cdot\hat{z}dxdy\Big|_{z=b\ surf.}}_{S_{31}}+\underbrace{\int_{x=-a}^{a}\int_{y=0}^{h}(\hat{x}-\hat{y})\cdot(-\hat{z}dxdy)\Big|_{z=b\ surf.}}_{S_{32}}+$$

$$+\underbrace{\int_{y=0}^{h}\int_{z=0}^{b}(\hat{x}-\hat{y})\cdot\hat{x}dydz\Big|_{x=-a}}_{S_{33}}+\underbrace{\int_{y=0}^{h}\int_{z=0}^{b}(\hat{x}-\hat{y})\cdot(-\hat{x}dydz)}_{S_{34}}+$$

$$+\int_{x=-a}^{a}\int_{z=0}^{b}(\hat{x}-\hat{y})\cdot(-\hat{y}dxdz)\Big|_{y=h}$$

$$=0+0+hb-hb+2ab=2ab$$

<u>Over Surface S_4</u>

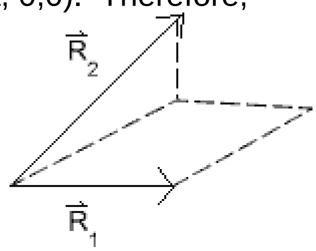
Similar to integrations over past surfaces. Therefore,

$$d\vec{S}_{42} = -\hat{z}dxdy \,\Big|_{z=b} \quad d\vec{S}_{43} = +\hat{z}dxdy \,\Big|_{z=0}$$

$$d\vec{S}_{44} = -\hat{x}dydz \,\Big|_{x=a} \quad dS_{41} = \hat{n}dS_{43}$$

where $\hat{n}$ is directed in the $-\hat{y}$, $+\hat{x}$ like directions.

To determine $\hat{n}$, consider the two vectors positioning points (-a, 0, b) and (a, h, 0) to point (-a, 0,0). Therefore,

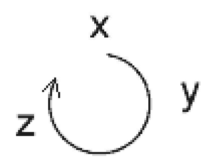


$\vec{R}_1 = b\hat{z}$

$\vec{R}_2 = 2a\hat{x} + h\hat{y}$

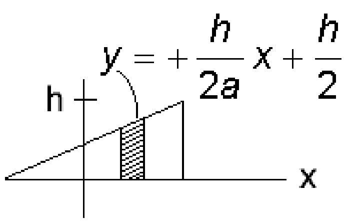
$\vec{R}_2 \times \vec{R}_1$ will give us the desired direction of $\hat{n}$

$$\hat{n} = \frac{\vec{R}_2 \times \vec{R}_1}{\left|\vec{R}_2 + \vec{R}_1\right|} = \frac{b\left[-2a\hat{y} + h\hat{x}\right]}{\left[(2ab)^2 + (bh)^2\right]^{1/2}}$$



$$\hat{n} = \frac{-2a\hat{y} + h\hat{x}}{\left[4a^2 + h^2\right]^{1/2}}$$

Recall that: $\vec{\nabla} \times \vec{F} = \hat{x} - \hat{y}$



$\therefore$

$$\iint_{S_4} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}_4 = \iint_{S_{41}} (\hat{x} - \hat{y}) \cdot \hat{n} dS_{41} \,\Big|_{on\ S_{41}} +$$

$$+ \int_{x=-a}^{a} \int_{y=0}^{[\frac{h}{2a}x + \frac{h}{2}]} (\hat{x} - \hat{y}) \cdot (-\hat{z}dxdy) \,\Big|_{z=b} + \underbrace{\int_{x=-a}^{a} \int_{y=0}^{[\frac{h}{2a}x + \frac{h}{2}]} (\hat{x} - \hat{y}) \cdot (\hat{z}dxdy) \,\Big|_{z=0}}_{S_{43}}$$

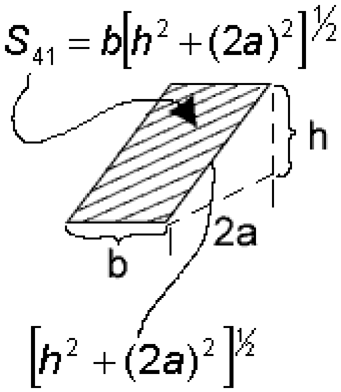
$$+ \underbrace{\int_{y=0}^{h} \int_{z=0}^{b} (\hat{x} - \hat{y}) \cdot (-\hat{x}dydz) \,\Big|_{x=a}}_{S_{44}}$$

$$\iint_{S_4} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}_4 = \int_{S_{41}}\int (\hat{x} - \hat{y}) \cdot \left(\frac{-2a\hat{y} + h\hat{x}}{\left[4a^2 + h^2\right]^{1/2}}\right) dS_{41} \,\Big|_{on\ S_{41}} - hb$$

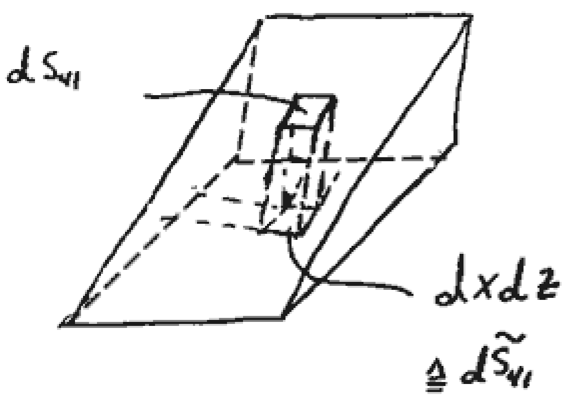
$$= -hb + \int_{S_{41}}\int \frac{2a + h}{[4a^2 + h^2]^{1/2}} dS_{41} \,\Big|_{on\ S_{41}}$$

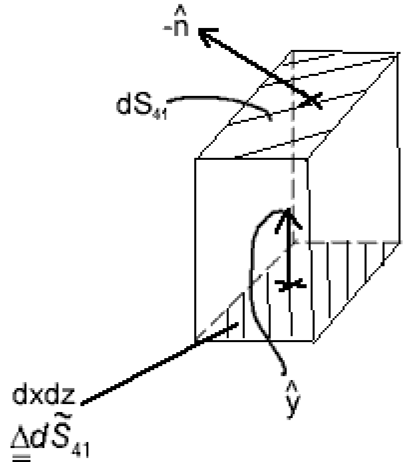
$$= -hb + \frac{2a + h}{[4a^2 + h^2]^{1/2}} \iint_{S_{41}} dS_{41} \Big|_{on\ S_{41}}$$

From standard geometry, it is easy to show that



Suppose we want to show the last integral from calculus. Consider projecting dS_{41} onto the y = 0 plane as shown below.





$$\left|d\tilde{S}_{41}\right| = \left|-\hat{n}\cdot\hat{y}\right| dS_{41} = dxdz$$

$\therefore$

$$dS_{41} = \frac{dxdz}{\left|-\hat{n}\cdot\hat{y}\right|} = \frac{dxdz}{\left(\dfrac{2a}{\left[4a^2+h^2\right]^{1/2}}\right)}$$

$\therefore$

$$\iint_{S_{41}} dS_{41} \Big|_{on\ S_{41}} = \int_{x=-a}^{a}\int_{z=0}^{b} \frac{\left[4a^2+h^2\right]^{1/2}}{2a} dxdz \Big|_{on\ S_{41}}$$

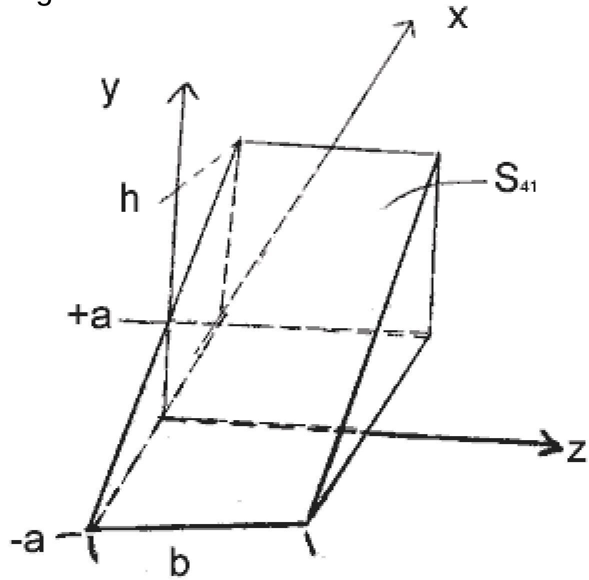
$$= b\left[4a^2+h^2\right]^{1/2}$$

Consequently,

$$\iint_{S_4} \left(\vec{\nabla}\times\vec{F}\right)\cdot d\vec{S}_4 = -hb + \frac{2a+h}{\left[4a^2+h^2\right]^{1/2}}\left(b\left[4a^2+h^2\right]^{1/2}\right)$$

Question: If the integrand was function of y, how must y change as x and z changes in order that one remains on S_{41}?

Consider the picture again.



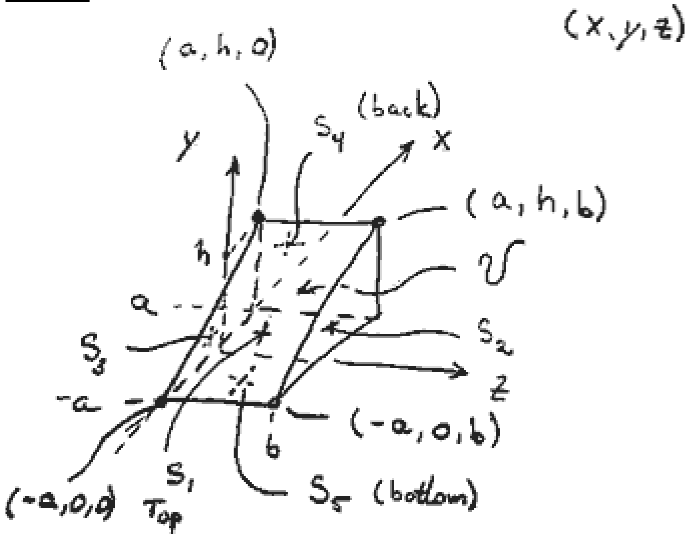
From the picture it is easy to see that y is independent of z. That is, holding x constant and varying z between 0 and b, y does not change. On the other hand, varying x changes y. From a simple equation of a line,

$$y = \frac{h}{2a}x + \frac{h}{2} \text{ for all z on } S_{41}$$

Therefore, if y appeared in the integrand, we would have to replace it by $y = \frac{h}{2a}x + \frac{h}{2}$ before we perform the integration.

5. Verifying the Divergence Theorem for the function
$\vec{F} = (x+y+z)\hat{y}$ in the volume V and over the surface S bounding V shown in the figure below.

<u>NOTE:</u> $d\vec{S}$ is the OUTWARD NORMAL surface element relative to V



Divergence Theorem

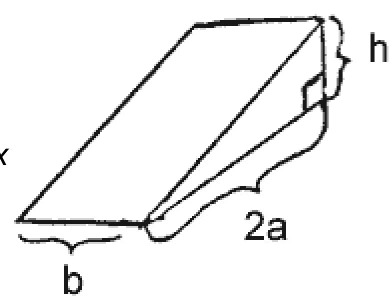
$$\oiint_S \vec{F}\cdot d\vec{s} = \iiint_V \left(\vec{\nabla}\cdot\vec{F}\right) dV$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z = 1$$

$$\oiint_S \vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot d\vec{S}_1 + \iint_{S_2}\vec{F}\cdot d\vec{S}_2 + \iint_{S_3}\vec{F}\cdot d\vec{S}_3 + \iint_{S_4}\vec{F}\cdot d\vec{S}_4 + \iint_{S_5}\vec{F}\cdot d\vec{S}_5$$

<u>Over Volume V</u>

$$\iiint_V \left(\vec{\nabla}\cdot\vec{F}\right) dV = 1\iiint_V dV = 1\int_{x=-a}^{a}\left[\int_{y=0}^{\frac{h}{2a}x+\frac{h}{2}}\left\{\int_{z=0}^{b} dz\right\} dy\right] dx$$

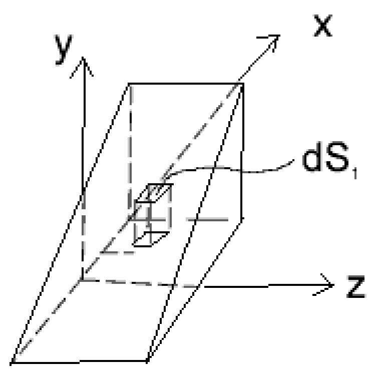


$$= b\int_{x=-a}^{a}\left[\frac{h}{2a}x + \frac{h}{2}\right] dx = b\left[\frac{hx^2}{4a} + \frac{h}{2}x\right]_{-a}^{a}$$

$$= bha$$

<u>Over Surface S_1 (Top)</u>

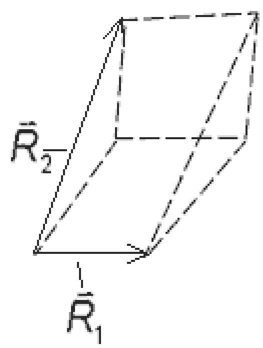
$$\iint_{S_1}\vec{F}\cdot d\vec{S}_1$$



In this problem, $d\vec{S}_1$ points in the $+\hat{y},-\hat{x}$ direction. Let $d\vec{S}_1 = \hat{n} dS_1$. To determine $\hat{n}$, consider the two vectors positioning points (-a, 0, b) and (a, h, 0) to point (-a, 0, 0,). Therefore,

$\vec{R}_1 = b\hat{z}$

$\vec{R}_2 = 2a\hat{x} + h\hat{y}$

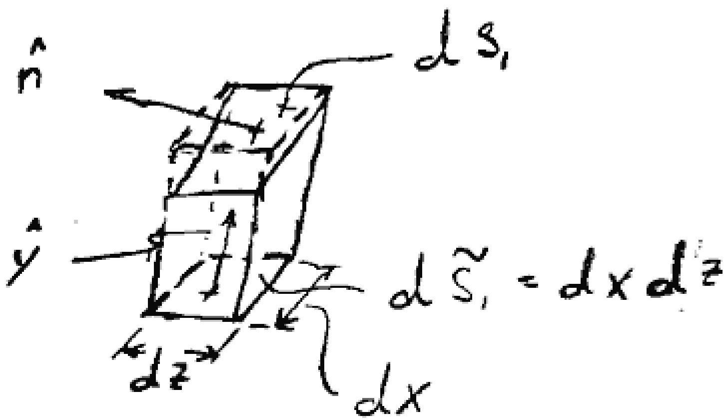


Therefore, $\vec{R}_1 \times \vec{R}_2$ will give us the desired direction of $\hat{n}$.

$$\hat{n} = \frac{\vec{R}_1 \times \vec{R}_2}{\left|\vec{R}_1 \times \vec{R}_2\right|} = \frac{b\left[2a\hat{y} - h\hat{x}\right]}{\left[(2ab)^2 + (bh)^2\right]^{1/2}}$$

$$\hat{n} = \frac{2a\hat{y} - h\hat{x}}{\left[4a^2 + h^2\right]^{1/2}}$$

Consider projecting dS_1 onto the y = 0 plane as shown in the figures above and below.



Projecting dS_1 onto $d\tilde{S}_1$ yields

$$d\tilde{S}_1 = |\hat{n}\cdot\hat{y}|dS_1 =$$

$$= dxdz$$

$$\therefore dS_1 = \frac{dxdz}{|\hat{n}\cdot\hat{y}|} = \frac{dxdz}{\left(\frac{2a}{[4a^2+h^2]^{1/2}}\right)}$$

$$dS_1 = \frac{[4a^2+h^2]^{1/2}}{2a}dxdz$$

$\therefore$

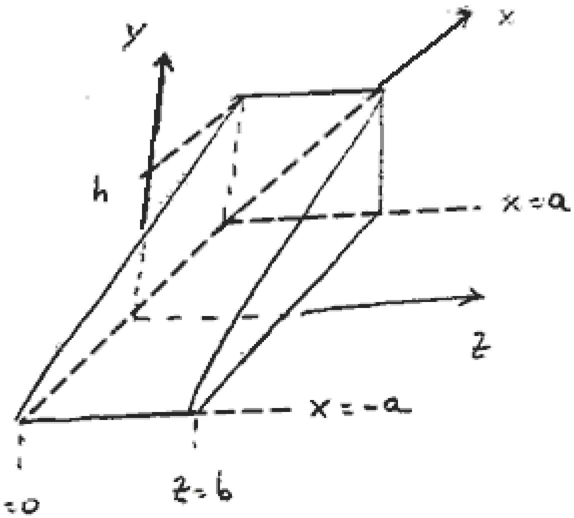
$$\iint_{S_1}\vec{F}\cdot d\vec{S}_1\Big|_{on\ S_1} = \iint_{S_1}\vec{F}\cdot\hat{n}dS_1\Big|_{on\ S_1} = \int_{\substack{x=-a\\ \tilde{S}_1}}^{a}\int_{z=0}^{b}\vec{F}\cdot\hat{n}\frac{[4a^2+h^2]^{1/2}}{2a}dxdz\Big|_{on\ S_1}$$

$$= \int_{x=-a}^{a}\int_{z=0}^{b}(x+y+z)\Big|_{on\ S_1}dxdz$$

$$\vec{F} = (x+y+z)\hat{y}$$

$$\hat{n} = \frac{2a\hat{y}-h\hat{x}}{[4a^2+h^2]^{1/2}}$$

Consider the picture again.



From the picture, it is easy to see that y is independent of z. That is, holding x constant and varying x between 0 and b, y does NOT change. On the other hand, varying x changes y. From a simple equation of a line,

$y = \frac{h}{2a}x + \frac{h}{2}$ for all z on S_1

that is,

$$y(x) = \frac{h}{2a}x + \frac{h}{2}$$

Thus,

$$\iint_{S_1} \vec{F} \cdot d\vec{S}_1 \Big|_{on\ S_1} = \int_{x=-a}^{a} \int_{z=0}^{b} (x + y + z) \Big|_{on\ S_1} dxdz = \int_{x=-a}^{a} \int_{z=0}^{b} \left[x + \frac{h}{2a}x + \frac{h}{2} + z \right] dxdz$$

$$= \int_{x=-a}^{a} \left[b\left(1 + \frac{h}{2a}\right)x + \left(\frac{b^2}{2} + \frac{hb}{2}\right) \right] dx$$

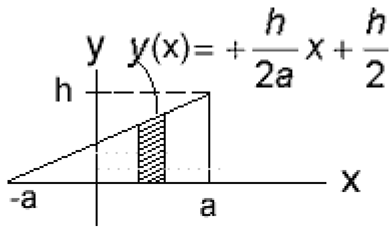
$$= \left[b\left(1 + \frac{h}{2a}\right)\frac{x^2}{2} + \left(\frac{b^2}{2} + \frac{hb}{2}\right)x \right|_{x=-a}^{a}$$

$$= 2\left(\frac{b^2}{2} + \frac{hb}{2}\right)a$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S}_1 \Big|_{on\ S_1} = (b + h)ab$$

<u>Over Surface S_2</u>

$$d\vec{S}_2 = +\hat{z}dxdy \Big|_{z=b}$$



$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 \Bigg|_{on\ S_2} = \int_{x=-a}^{a} \int_{y=0}^{[\frac{h}{2a}x+\frac{h}{2}]} \left[(x+y+z)\hat{y}\right] \cdot \left[+\hat{z}dxdy\right] \Bigg|_{z=b}$$

$= 0$

<u>Over Surface S_3</u>

$$\iint_{S_3} \vec{F} \cdot d\vec{S}_3$$

$d\vec{S}_3 = -\hat{z}dxdy$

By analogy to S_2,

$$\iint_{S_3} \vec{F} \cdot d\vec{S}_3 = \int_{x=-a}^{a} \int_{y=0}^{[\frac{h}{2a}x+\frac{h}{2}]} \left[(x+y+z)\hat{y}\right] \cdot \left[-\hat{z}dxdy\right] \Bigg|_{z=0} = 0$$

<u>Over Surface S_4</u> (back)

$$\iint_{S_4} \vec{F} \cdot d\vec{S}_4$$

$d\vec{S}_4 = \hat{x}dydz \Big|_{at\ x=a}$

But $\vec{F} \cdot \hat{x} = 0 \therefore$

$$\iint_{S_4} \vec{F} \cdot d\vec{S}_4 = 0$$

<u>Over Surface S_5</u>

$$\iint_{S_5} \vec{F}\cdot d\vec{S}_5$$

$$d\vec{S}_5 = -\hat{y}dxdz\Big|_{y=0}$$

$$\therefore$$

$$\iint_{S_5} \vec{F}\cdot d\vec{S} = \int_{x=-a}^{a}\int_{z=0}^{b}\left[(x+y+z)\hat{y}\right]\cdot\left[-\hat{y}dxdz\right]\Big|_{y=0}$$

$$= -\int_{x=-a}^{a}\int_{z=0}^{b}\left[x+0+z\right]dxdz$$

$$= -\left\{[\frac{bx^2}{2}\Big|_{-a}^{a} + 2a\frac{z^2}{2}\Big|_0^b\right\} = -ab^2$$

Consequently,

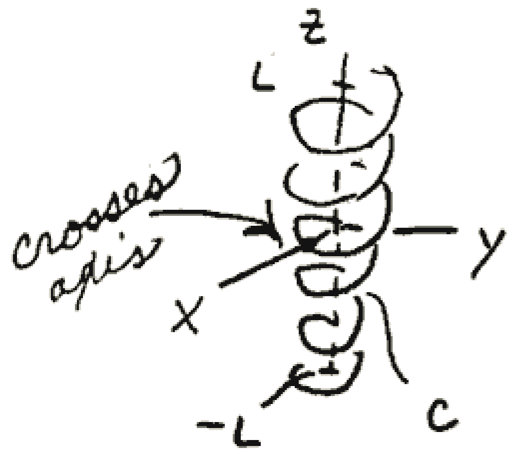
$$\oiint_S \vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot d\vec{S}_1 + \iint_{S_2}\vec{F}\cdot d\vec{S}_2 + \iint_{S_3}\vec{F}\cdot d\vec{S}_3 + \iint_{S_4}\vec{F}\cdot d\vec{S}_4 + \iint_{S_5}\vec{F}\cdot d\vec{S}_5$$

$$= (b+h)ab + 0 + 0 + 0 - ab^2$$

$$= abh$$

$$= \iiint_V (\vec{\nabla}\cdot\vec{F})dV$$

6. Evaluate the vector $\vec{F} = A\hat{\varphi}(\varphi)$ over the coil illustrated below where r=R and $\varphi = {}^{+}az$. The coil extends form –L to +L.



Employ a cylindrical coordinate system.

$d\vec{l} = dr\hat{r}(\varphi) + rd\varphi\hat{\varphi}(\varphi) + dz\hat{z}$

$dL = \left[d\vec{l} \cdot d\vec{l}\right]^{1/2} = \left[(dr)^2 + (rd\varphi)^2 + (dz)^2\right]^{1/2}$

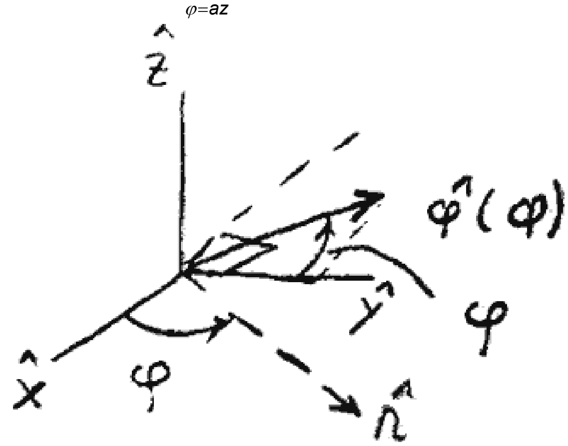
But r=R a constant $\therefore$ dr=0

$\varphi = az \quad \therefore \quad d\varphi = +adz$

$\therefore$

$dL = \left[(ra)^2 + 1\right]^{1/2} dz$

$$\int_c \vec{F} dL \Big|_{on\ coil} = \int_{\substack{-L \\ z=}}^{L} A\hat{\varphi}(\varphi)\left[(ra)^2 + 1\right]^{1/2} dz \Big|_{\substack{r=R \\ \varphi=az}}$$



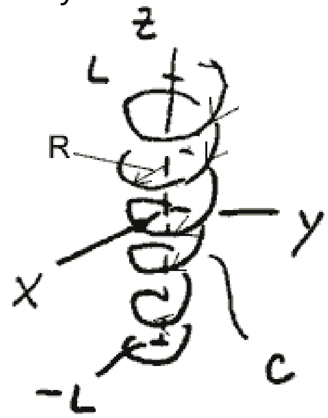
$\hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y}$

But, $\varphi = az$

$$\int_c \vec{F} dL \Big|_{on\ coil} = \int_{z=-L}^{L} A\left[-\sin(az)\hat{x} + \cos(az)\hat{y}\right]\left[(Ra)^2 + 1\right]^{1/2} dz$$

$$= +A\left[(Ra)^2 + 1\right]^{1/2}\left[-\hat{x}\int_{-L}^{L}\sin(az)dz + \hat{y}\int_{-L}^{L}\cos(az)dz\right]$$

7. Evaluate the vector $\vec{F} = A\hat{\varphi}(\varphi)$ along the coil illustrated below where r=R and $\varphi = az$. The coil extends from –L to L.
Employ a cylindrical coordinate system.



$$\int_c \vec{F} \cdot d\vec{l}$$

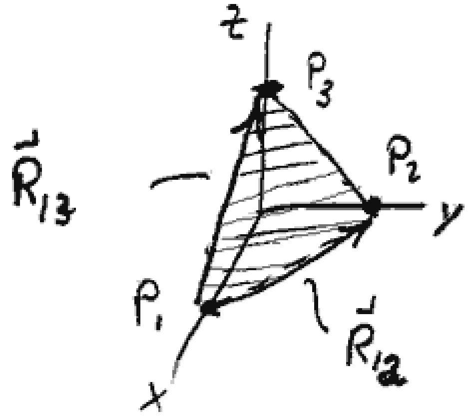
$$d\vec{l} = dr\hat{r}(\varphi) + rd\varphi\hat{\varphi}(\varphi) + dz\hat{z}$$

$$= Radz\hat{\varphi}(\varphi) + dz\hat{z}\Big|_{\substack{r=R \\ \varphi=az}}$$

$$\int_{\substack{+L \\ z=}}^{-L} \left[A\hat{\varphi}(\varphi)\right] \cdot \left[Ra\ dz\ \hat{\varphi}(\varphi) + dz\ \hat{z}\right] =$$

$$= \int_{z=+L}^{-L} ARadz$$

8. A plane bounded by the first octant passes through points $P_1(x_1, 0, 0)$, $P_2(0, y_2, 0)$, and $P_3(0, 0, z_3)$. Determine the unit vector normal to the surface of the plane.



Consider the two vectors which span the plane; $\vec{R}_{13}$ and $\vec{R}_{12}$. Then, by the cross product, the vector normal to the plane at point 1 is

$$\vec{R}_n = \vec{R}_{13} \times \vec{R}_{12}$$

or

$$\vec{R}_n = \vec{R}_{12} \times \vec{R}_{13} = -(\vec{R}_{13} \times \vec{R}_{12}) \qquad \text{(plane has two sides)}$$

The unit vector is then

$$\hat{n} = \pm \frac{\vec{R}_{13} \times \vec{R}_{12}}{\left|\vec{R}_{13} \times \vec{R}_{12}\right|} \qquad \vec{R}_1 = x_1\hat{x}, \quad \vec{R}_2 = y_2\hat{y}, \quad \vec{R}_3 = z_3\hat{z}$$

$$\vec{R}_{13} = -x_1\hat{x} + z_3\hat{z}$$

$$\vec{R}_{12} = -x_1\hat{x} + y_2\hat{y} \qquad \text{Look at picture for signs}$$

$$\vec{R}_{13} \times \vec{R}_{12} = -x_1 y_2\hat{z} - x_1 z_3\hat{y} - y_2 z_3\hat{x}$$

$$\left|\vec{R}_{13} \times \vec{R}_{12}\right| = \left[(\vec{R}_{13} \times \vec{R}_{12}) \cdot (\vec{R}_{13} \times \vec{R}_{12})\right]^{1/2}$$

$$= \left[(y_2 z_3)^2 + (x_1 z_3)^2 + (x_1 y_2)^2\right]^{1/2}$$

$\therefore$

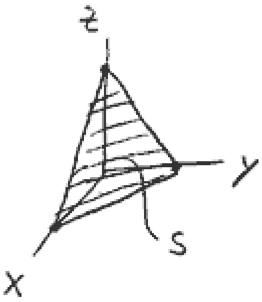
$$\hat{n} = -\frac{\left[y_2 z_3\hat{x} + x_1 z_3\hat{y} + x_1 y_2\hat{z}\right]}{\left[(y_2 z_3)^2 + (x_1 z_3)^2 + (x_1 y_2)^2\right]^{1/2}}$$

or

$$\hat{n} = +\frac{\left[y_2 z_3\hat{x} + x_1 z_3\hat{y} + x_1 y_2\hat{z}\right]}{\left[(y_2 z_3)^2 + (x_1 z_3)^2 + (x_1 y_2)^2\right]^{1/2}}$$

x_1, y_2, and z_3 are given numbers.
Since the surface is a plane, $\hat{n}$ is the same at each point on the surface!

9. Reconsider problem 8. Evaluate the vector field $\vec{F} = xy\hat{z} + zx\hat{y} + x\hat{z}$ passing normal through the plane bounded by the first octant passing through points (x_1, 0, 0), (0, y_2, 0) and (0, 0, z_3). Note, x_1, y_2 and z_3 are numbers. Choose that surface normal to point in the $+\hat{x}$ like direction.



From problem 1,

$$\hat{n} = \frac{\left[y_2 z_3 \hat{x} + x_1 z_3 \hat{y} + x_1 y_2 \hat{z}\right]}{\left[(y_2 z_3)^2 + (x_1 z_3)^2 + (x_1 y_2)^2\right]^{1/2}}$$

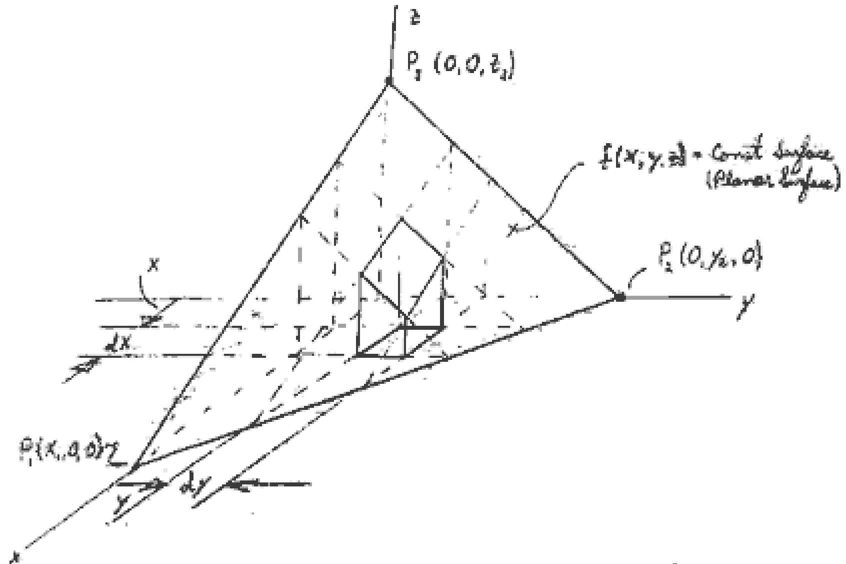
$$\iint_S \vec{F} \cdot d\vec{s} =$$

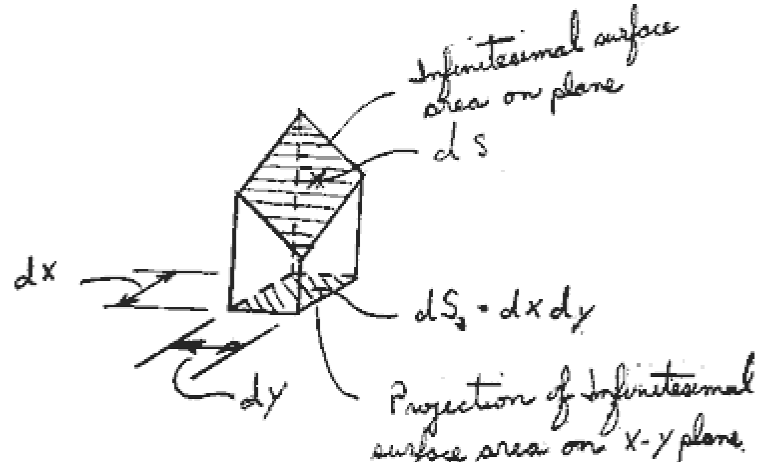
guarantees that only normal component of $\vec{F}$ relative to S is evaluated

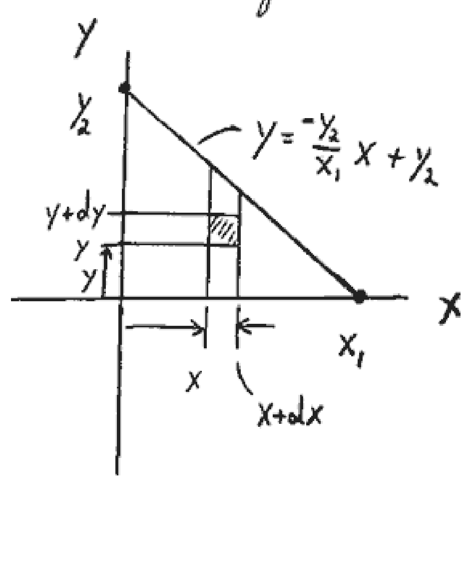
$$= \iint_S \left[xy\hat{x} + xz\hat{y} + x\hat{z}\right] \cdot \left[\frac{y_2 z_3 \hat{x} + x_1 z_3 \hat{y} + x_1 y_2 \hat{z}}{\left[(y_2 z_3)^2 + (x_1 z_3)^2 + (x_1 y_2)^2\right]^{1/2}}\right] dS$$

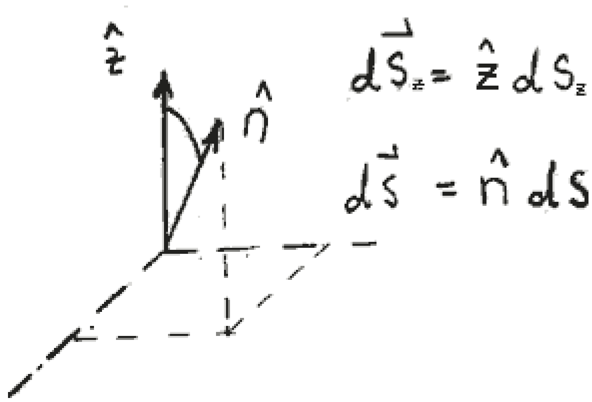
$$= \frac{1}{\left[(y_2 z_3)^2 + (x_1 z_3)^2 + (x_1 y_2)^2\right]^{1/2}} \iint_S \left[xyy_2 z_3 + xzx_1 z_3 + xx_1 y_2\right] dS$$

Note that x, y and z exist in the integrand. Now, let us project the surface element dS onto the x-y plane and integrate over the projected image in this plane. Refer to the figures below.


z
P_3 (0,0,z_3)
f(x,y,z) = Const Surface
(Planar Surface)
P_2 (0,y_2,0)
y
x
dx
P_1(x_1,0,0)
y
dy
x


Infinitesimal surface
area on plane
dS
dx
dS_3 = dx dy
dy
Projection of Infinitesimal
surface area on x-y plane.


y
y_2
$y = \frac{-y_2}{x_1} x + y_2$
y+dy
y
y
x
x_1
x
x+dx



Projecting dS onto dS_z yields

$dS_z = (\hat{z} \cdot \hat{n}) dS$

$\therefore$

$$dS_z = \frac{x_1 y_2}{\left[(y_2 z_3)^2 + (x_1 z_3)^2 + (x_1 y_2)^2\right]^{1/2}} dS = dxdy$$

$\therefore$

$$dS = \frac{\left[(y_2 z_3)^2 + (x_1 z_3)^2 + (x_1 y_2)^2\right]^{1/2}}{x_1 y_2} dxdy$$

From the figure above, the limits of integration for x are 0 and x_1. At any $0 \le x \le x_1$, the range of y is $0 \le y \le \left(\frac{-y_2}{x_1} x + y_2 \right)$.

To complete our analysis, we must determine how z is related to x and y in order that we remain on the planar surface described. Two special cases can be easily obtained in the y=0 and z=0 planes; equations of lines.

For y=0, $z = \frac{-z_3}{x_1} x + z_3$

For z=0, $y = \frac{-y_2}{x_1} x + y_2$

Since the surface is planar, x, y and z are linearly related to each other. Therefore,

f(x, y, z) = $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = K$ = a constant

Let K=1. In order that the two equations of a line be satisfied in this expression we force

$a = x_1$

$b = y_2$

$c = z_3$

$\therefore$

f(x, y, z)= $\frac{x}{x_1}+\frac{y}{y_2}+\frac{z}{z_3}=1$

This expression correctly describes our plane. Since x and y are being varied in the integration, z must appropriately change in order to remain on the surface. Therefore,

$$z(x,y)=z_3-\frac{z_3}{x_1}x-\frac{z_3}{y_2}y$$

Now, let us combine all our terms.

$$\iint_S \vec{F}\cdot d\vec{S}=\frac{1}{\left[(y_2z_3)^2+(x_1z_3)^2+(x_1y_2)^2\right]^{1/2}}\iint_S\left[xyy_2z_3+xzx_1z_3+xx_1y_2\right]dS\,\Big|_{\text{on Surface } S}$$

$$=\frac{\left[(y_2z_3)^2+(x_1z_3)^2+(x_1y_2)^2\right]^{1/2}}{x_1y_2\left[(y_2z_3)^2+(x_1z_3)^2+(x_1y_2)^2\right]^{1/2}}\iint_{S_z}\left[xyy_2z_3+xzx_1z_3+xx_1y_2\right]dxdy\,\Big|_{\text{on } S}$$

$$=\frac{1}{x_1y_2}\int_0^{x_1}\int_0^{\left(-\frac{y_2}{x_1}x+y_2\right)}\left[xyy_2z_3+xzx_1z_3+xx_1y_2\right]\Big|_{z=z(x,y)=z_3-\frac{z_3}{x_1}x-\frac{z_3}{y_2}y}dxdy$$

$\therefore$

$$=\frac{1}{x_1y_2}\int_0^{x_1}\left\{\int_0^{\left(-\frac{y_2}{x_1}x+y_2\right)}\left[xyy_2z_3+x\left(z_3-\frac{z_3}{x_1}x-\frac{z_3}{y_2}y\right)x_1z_3+xx_1y_2\right]dy\right\}dx$$

The remainder of this problem is simple integral mechanics and is left for the student to solve.